Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

Topology and its Applications 135 (2004) 207–213

Topology and its Applications

www.elsevier.com/locate/topol

An uncountable family of metric compactifications of the ray with remainder pseudo-arc

Verónica Martínez-de-la-Vega

Instituto de Matemáticas, Departamento de Matemáticas, UAM – Iztapalapa, UNAM, Circuito Exterior, Cd. Universitaria, México, DF 04510, Mexico

Received 22 November 2001; received in revised form 28 January 2003

Abstract

In this paper we construct an uncountable family of metric compactifications of the ray with the remainder being the pseudo-arc, answering a question posed by Marwan M. Awartani.

MSC: primary 54F15; secondary 54F50, 54F65, 54C15

Keywords: Continuum; Pseudo-arc; Compactification

1. Introduction and preliminaries

On 1993 M. Awartani constructed an uncountable collection of mutually incomparable chainable continua [1]. On 1995 he asked if all compactifications of the ray with remainder being the pseudo-arc were homeomorphic or not. This question was answered in the negative in [6] where two non-homeomorphic compactifications of the ray with remainder being the pseudo-arc were constructed. In this paper an uncountable family of such compactifications is constructed.

A *continuum* is a compact, connected metric space. Throughout this paper the letter X denotes a continuum with a metric d. A *map* is a continuous function. Given two points a, b in an arc L, ab denotes the subarc in L joining them.

A finite sequence $\mathcal{C} = \{U_1, \dots, U_s\}$ of (not necessarily open) subsets of X is said to be a *chain* provided that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. The sets U_i are called *links*, U_1 is the *first link* and U_s is the *last link*. We denote $\overline{\mathcal{C}} = \{\mathrm{cl}_X(U_1), \dots, \mathrm{cl}_X(U_s)\}$

E-mail address: vmvm@matem.unam.mx (V. Martínez-de-la-Vega).

0166-8641/$ – see front matter
doi:10.1016/S0166-8641(03)00164-0

and $\mathcal{C}^* = U_1 \cup \cdots \cup U_s$. If each link U_i is connected, $\mathcal{C}$ is called a *connected chain*. Chain $\mathcal{C}$ is called ε-*chain* provided that $\operatorname{diam}(U_i) < \varepsilon$, for each $i \in \{1, \dots, s\}$. Given a chain $\mathcal{C} = \{U_1, \dots, U_s\}$, a *subchain* is a chain of the form $\mathcal{C}_{(i,j)} = \{U_i, U_{i+1}, \dots, U_j\}$.

Let $\mathcal{C} = \{U_1, \dots, U_s\}$ and $\mathcal{D} = \{V_1, \dots, V_t\}$ be chains. We say that $\mathcal{D}$ *refines* $\mathcal{C}$ provided that, for each $j \in \{1, \dots, t\}$ there exists $i \in \{1, \dots, s\}$ such that $V_j \subset U_i$. Chain $\mathcal{D}$ *is crooked in* chain $\mathcal{C}$ if $\mathcal{D}$ refines $\mathcal{C}$ and for any indices i, j, m and n with $V_i \cap U_m \neq \emptyset$, $V_j \cap U_n \neq \emptyset$ and $m < n - 2$, there exist indices k and l with $i < k < l < j$ (or $i > k > l > j$) and $V_k \subset U_{n-1}$, $V_l \subset U_{m+1}$.

The continuum X is *chainable* if for each $\varepsilon > 0$ there exists an ε-chain $\mathcal{C}$ such that $X = \mathcal{C}^*$.

A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. A continuum which is not decomposable is said to be *indecomposable*.

A property of a continuum X is said to be *hereditary* provided that each subcontinuum of X has the property.

The *pseudo-arc* is a nondegenerate, hereditarily indecomposable, chainable continuum [5, p. 28]. Bing has proved that there is a unique continuum satisfying these conditions (see [2]).

Let $\mathcal{C}$ be a chain in the plane, an arc L is said to *go straight in* $\mathcal{C}$ provided that $L \subset \mathcal{C}^*$ and if two points a and b are in $L \cap U$, for some link U of $\mathcal{C}$, then the subarc ab of L is contained in U. Notice that, if J is a subarc of L and $\mathcal{C}_{(i,j)}$ is a subchain of $\mathcal{C}$ such that $J \subset \mathcal{C}^*_{(i,j)}$, then J goes straight in $\mathcal{C}_{(i,j)}$.

An arc L is *crooked* in a chain in the plane $\mathcal{C} = \{U_1, \dots, U_s\}$ if $L \subset \mathcal{C}^*$ and each time that there exist points $p, q \in L$ such that $p \in U_m$, $q \in U_n$ and $m < n - 2$, then there exist points $p', q' \in pq$ such that $p < p' < q' < q$ (in the natural order of the arc pq) and $p' \in U_{n-1}$ and $q' \in U_{m+1}$. Notice that, if J is a subarc of L, L is crooked in $\mathcal{C}$ and $\mathcal{C}(i, j)$ is a subchain of $\mathcal{C}$ such that $J \subset \mathcal{C}(i, j)^*$, then J is crooked in $\mathcal{C}(i, j)$.

The following results are easy to prove.

Lemma 1. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be chains such that $\mathcal{A}$ refines $\mathcal{B}$ and $\mathcal{B}$ is crooked in $\mathcal{C}$, then $\mathcal{A}$ is crooked in $\mathcal{C}$.*

Lemma 2. *Let $\mathcal{B}$ and $\mathcal{C}$ be chains and L an arc such that L goes straight in $\mathcal{B}$ and $\mathcal{B}$ is crooked in $\mathcal{C}$. Then L is crooked in $\mathcal{C}$.*

Construction 3. *The pseudo-arc can be constructed as follows* (*see* [5, p. 28, (1.7)]):

Let $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be a sequence of connected chains in the plane such that for distinct points p_0 and q_0,

(1) *each $\mathcal{C}_n$ has p_0 in its first link and q_0 in its last link,*
(2) *$\mathcal{C}_n$ is a $\frac{1}{2^n}$-chain for each n,*
(3) *$\overline{\mathcal{C}_{n+1}}$ refines $\mathcal{C}_n$ for each n, and*
(4) *$\mathcal{C}_{n+1}$ is crooked in $\mathcal{C}_n$ for each n.*

Then $\mathcal{P} = \bigcap\{\mathcal{C}_n^*\colon n \in \mathbb{N}\}$ is a pseudo-arc.

Construction 4. *Consider the sequence of chains $\{\mathcal{C}_n\}_{n=1}^{\infty}$ as in Construction 3, we also ask that $p_0 = (0,0)$, $q_0 = (1,0)$, the links of each $\mathcal{C}_n$ are convex and $\mathcal{C}_1^* \subset [0,1] \times [-1,1]$. For each $n \in \mathbb{N}$ we consider an arc L_n such that L_n joins p_0 and q_0 and L_n goes straight in $\mathcal{C}_n$.*

It is easy to check that $\lim L_n = \mathcal{P}$ (with the Hausdorff metric, see [4, p. 11, 2.1]).

Lemma 5. *For each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $n < m_n$ and, if $k \geqslant m_n$, L is a subarc of L_n and $h : L \to L_k$ is an embedding, then one of the following condition holds*:

(a) $\operatorname{diam}(h(L)) \leqslant \frac{1}{4}$, *or*
(b) *there exist points u, v in L such that $\|u - v\| < \frac{1}{n}$ and $\|h(u) - h(v)\| \geqslant \frac{1}{8}$.*

Proof. Consider chain $C_n = \{U_1, U_2, \dots, U_s\}$ as in Construction 3.

Let $M \in \mathbb{N}$ be such that $\frac{3s+4}{2^M} < \frac{1}{4}$ and $\frac{s}{2^M} + \frac{1}{2^M} < \frac{1}{16}$.

Take $m_n = M + 1$, $k \geqslant m_n$, $h : L \to L_k$ an embedding and chain $\mathcal{C}_M = \{V_1, \dots, V_r\}$ as in Construction 3.

Assume that $\operatorname{diam}(h(L)) > \frac{1}{4}$, then there exist points p and q of $h(L)$ such that $\|p - q\| > \frac{1}{4}$. Consider the subarc pq of $h(L)$. Since $pq \subset L_k$, L_k goes straight in $\mathcal{C}_k$ and $\mathcal{C}_k$ is crooked in $\mathcal{C}_M$, pq is crooked in $\mathcal{C}_M$. Thus there exists a minimal subchain $\mathcal{C}_M(i, j)$ of $\mathcal{C}_M$ such that pq is crooked in $\mathcal{C}_M(i, j)$, where $i \leqslant j$.

Claim 1. $j - i > 3s + 3$.

Proof. In order to prove Claim 1, suppose to the contrary that $j - i \leqslant 3s + 3$. Then $\frac{1}{4} < \|p - q\| \leqslant \operatorname{diam}(pq) \leqslant \operatorname{diam}(V_i \cup \dots \cup V_j) \leqslant \operatorname{diam}(V_i) + \dots + \operatorname{diam}(V_j) \leqslant \frac{j-i+1}{2^M} \leqslant \frac{3s+4}{2^M} < \frac{1}{4}$. This is a contradiction that proves Claim 1. $\square$

Claim 2. $\operatorname{diam}(V_{i+s} \cup \dots \cup V_{j-s-1}) > \frac{1}{8} + \frac{2}{2^M}$.

Proof. In order to prove Claim 2, notice that, by the choice of M,

$$\operatorname{diam}(V_i \cup \dots \cup V_{i+s-1}) + \operatorname{diam}(V_{j-s} \cup \dots \cup V_j) \leqslant \frac{s-1}{2^M} + \frac{s}{2^M} < \frac{1}{8} - \frac{2}{2^M}.$$

Since

$$\begin{aligned}\frac{1}{4} &< \|p - q\| \leqslant \operatorname{diam}(pq) \leqslant \operatorname{diam}(V_i \cup \dots \cup V_j)\\ &\leqslant \frac{1}{8} - \frac{2}{2^M} + \operatorname{diam}(V_{i+s} \cup \dots \cup V_{j-s-1}),\end{aligned}$$

we have that

$$\frac{1}{8} + \frac{2}{2^M} < \operatorname{diam}(V_{i+s} \cup \dots \cup V_{j-s-1}). \qquad \square$$

Claim 3. *$h(L)$ contains $s+1$ pairwise disjoint arcs $A_1,\dots,A_{s+1}$ with* $\operatorname{diam}(A_t)\geqslant\frac{1}{8}$, *for each $t\in\{1,\dots,s+1\}$.*

Proof. We prove Claim 3. Fix an order $<$ in the arc pq in such a way that $p<q$. By the choice of i and j, $pq\cap V_i\neq\emptyset$, $pq\cap V_j\neq\emptyset$. Thus, we can choose points $p_1\in pq\cap V_i$ and $q_{s+1}\in pq\cap V_j$.

By Claim 1, $i<j-2$. Since pq is crooked in $\mathcal{C}_M$, there exist points $q_1,p_2\in p_1q_{s+1}$ such that $p_1<q_1<p_2<q_{s+1}$, $q_1\in V_{j-1}$ and $p_2\in V_{i+1}$.

By Claim 1, $i+1<j-2$. Since pq is crooked in $\mathcal{C}_M$, there exist points $q_2,p_3\in p_2q_{s+1}$ such that $p_2<q_2<p_3<q_{s+1}$, $q_2\in V_{j-1}$ and $p_3\in V_{i+2}$.

Since, by Claim 1, $i+s-1<j-2$, it is possible to continue this procedure to show that there exist points $p_1,p_2,\dots,p_{s+1},q_1,q_2,\dots,q_{s+1}$ in pq such that:

(a) $p_1<q_1<p_2<q_2<\cdots<p_s<q_s<p_{s+1}<q_{s+1}$,
(b) $p_t\in V_{i+t-1}$ and $q_t\in V_{j-1}$ for each $t\in\{1,\dots,s+1\}$.

For each $t\in\{1,\dots,s+1\}$, let $A_t=p_tq_t$. Clearly, $A_1,\dots,A_{s+1}$ are pairwise disjoint subarcs of pq. Since A_t is a connected set and intersects V_{i+t-1} and V_{j-1} and $i+t-1\leqslant i+s$, A_t intersects each one of the links $V_{i+s},\dots,V_{j-s-1}$. Thus, given points $a,b\in V_{i+s}\cup\cdots\cup V_{j-s-1}$, there exist points $c,e\in A_t$ such that $\|a-c\|,\|b-e\|<\frac{1}{2^M}$. Then $\operatorname{diam}(V_{i+s}\cup\cdots\cup V_{j-s-1})\leqslant\operatorname{diam}(A_t)+\frac{2}{2^M}$. By Claim 2, $\frac{1}{8}\leqslant\operatorname{diam}(A_t)$.

Claim 3 has been proved. □

We are ready to finish the proof of the lemma.

Since the arc L_n goes straight in $\mathcal{C}_n$ and $h^{-1}(pq)$ is a subarc of L_n, there exist points $u_0,\dots,u_m\in pq$ such that $m\leqslant s$, $p=u_0<u_1<\cdots<u_m=q$ and, for each $t\in\{1,\dots,m\}$, $h^{-1}(u_{t-1}u_t)\subset U_l$ for some $l\in\{1,\dots,s\}$. By the box principle, since $\{A_1,\dots,A_{s+1}\}$ are disjoint, there exists $j_0\in\{1,\dots,s+1\}$ such that A_{j_0} does not contain any point u_t. Thus, there exists $t_0\in\{1,\dots,m\}$ such that $A_{j_0}\subset u_{t_0-1}u_{t_0}$. Then $\operatorname{diam}(h^{-1}(A_{j_0}))\leqslant\frac{1}{2^n}$ and $\operatorname{diam}(A_{j_0})\geqslant\frac{1}{8}$. Therefore, there exist points u,v in $h^{-1}(A_{j_0})\subset L$ such that

$$\|u-v\|<\frac{1}{2^n}<\frac{1}{n}\quad\text{and}\quad\big\|h(u)-h(v)\big\|\geqslant\frac{1}{8}.$$

This completes the proof of the lemma. □

For each $n\in\mathbb{N}$, let $\mathcal{F}_n=\{\alpha:L_n\to[0,\frac{1}{32}]\mid\alpha$ is a map and $|\alpha(p)-\alpha(q)|\leqslant\|p-q\|$ for every $p,q\in L_n\}$. Given $\alpha\in\mathcal{F}_n$, let $J_\alpha=\{(p,\alpha(p))\in\mathbb{R}^3\colon\ p\in L_n\}$.

Lemma 6. *For each $n\in\mathbb{N}$, let $m_n\in\mathbb{N}$ be as in Lemma* 5*. Suppose that $k\geqslant m_n$, $\alpha\in\mathcal{F}_n$, $\beta\in\mathcal{F}_k$, J is a subarc of J_α and $h:J\to J_\beta$ is an embedding. Then one of the following condition holds*:

(a) $\operatorname{diam}(h(J))\leqslant\frac{1}{3}$, *or*
(b) *there exist points u,v in J such that $\|u-v\|<\frac{2}{n}$ and $\|h(u)-h(v)\|\geqslant\frac{1}{8}$.*

Proof. Let $f : L_n \to J_\alpha$ be given by $f(p) = (p, \alpha(p))$ and $g : L_k \to J_\beta$, where $g(q) = (q, \beta(q))$. Notice that f and g are homeomorphisms, $\|(p,0) - f(p)\| \leqslant \frac{1}{32}$ for each $p \in L_n$ and $\|(q,0) - g(q)\| \leqslant \frac{1}{32}$ for each $q \in L_k$.

Notice that $\|p - w\| \leqslant \|f(p) - f(w)\| \leqslant 2\|p - w\|$ for every $p, w \in L_n$ and $\|q - z\| \leqslant \|g(q) - g(z)\| \leqslant 2\|q - z\|$ for every $q, z \in L_k$.

Let $L = f^{-1}(J)$ and $h' : L \to L_k$ be given by $h' = g^{-1} \circ h \circ (f|L)$. Then L is a subarc of L_n and h' is an embedding. Thus we can apply Lemma 5 and we obtain that one of the following condition holds:

(a′) $\operatorname{diam}(h'(L)) \leqslant \frac{1}{4}$, or
(b′) there exist points u', v' in L such that $\|u' - v'\| < \frac{1}{n}$ and $\|h'(u') - h'(v')\| \geqslant \frac{1}{8}$.

Suppose first that $\operatorname{diam}(h'(L)) \leqslant \frac{1}{4}$. Given $(p, \alpha(p))$, $(w, \alpha(w)) \in J$, since $L = f^{-1}(J)$, $p, w \in L$. Thus $\|h'(p) - h'(w)\| \leqslant \frac{1}{4}$. Since $\|g(h'(p)) - h'(p)\| \leqslant \frac{1}{32}$ and $\|g(h'(w)) - h'(w)\| \leqslant \frac{1}{32}$, $\|g(h'(p)) - g(h'(w))\| \leqslant \frac{1}{3}$.
Hence, $\|h(p, \alpha(p)) - h(w, \alpha(w))\| \leqslant \frac{1}{3}$.
We have shown that $\operatorname{diam}(h(J)) \leqslant \frac{1}{3}$.

Now, suppose that there exist points u', v' in L such that $\|u' - v'\| < \frac{1}{n}$ and $\|h'(u') - h'(v')\| \geqslant \frac{1}{8}$. Let $u = f(u')$ and $v = f(v')$. Then $\|u - v\| < \frac{2}{n}$ and $\|g(h'(u')) - g(h'(v'))\| \geqslant \frac{1}{8}$. Thus, $\|h(u) - h(v)\| \geqslant \frac{1}{8}$.

This ends the proof of the lemma. □

Construction 7. *First, we construct, inductively, a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers. For each $n \in \mathbb{N}$ consider a number m_n as in Lemma* 5. *Let $n_1 = 1$ and, for each $k \in \mathbb{N}$, let $n_{k+1} = m_{n_k}$. Notice that $n_1 < n_2 < \cdots$. Given an infinite subset T of $\{n_1, n_2, \ldots\}$, we construct a compactification $\mathcal{S}_T$ of the ray $[0, \infty)$ in the following way.*

Suppose that $T = \{j_1, j_2, \ldots\}$, where $j_1 < j_2 < \cdots$. For each $i \in \mathbb{N}$, let $\alpha(j_i) : L_{j_i} \to [0, \frac{1}{32}]$ be given by

$$\alpha(j_i)((x,y)) = \begin{cases} x\left(\dfrac{1}{2^{i+5}}\right) + (1-x)\left(\dfrac{1}{2^{i+4}}\right), & \text{if } i \text{ is odd,} \\[2ex] x\left(\dfrac{1}{2^{i+4}}\right) + (1-x)\left(\dfrac{1}{2^{i+5}}\right), & \text{if } i \text{ is even.} \end{cases}$$

Notice that $\alpha(j_i) \in \mathcal{F}_{j_i}$. Let $J(j_i) = J_{\alpha(j_i)} = \{(p, \alpha(j_i)(p)) \in \mathbb{R}^3 \colon p \in L_{j_i}\}$.
Let $\mathcal{R}_T = \bigcup\{J(j_i) \colon i \in \mathbb{N}\}$. Finally, let $\mathcal{S}_T = \operatorname{cl}_{\mathbb{R}^2}(\mathcal{R}_T)$.

The following lemma is easy to prove.

Lemma 8. *Given an infinite subset T of $\{n_1, n_2, \ldots\}$, conditions* (a)–(d) *follow.*

(a) *$\mathcal{R}_T$ is homeomorphic to $[0, \infty)$,*
(b) *$\lim J(j_i) = \mathcal{P}$ (with the Hausdorff metric),*

(c) $\mathcal{S}_T = \mathcal{P} \cup \mathcal{R}_T$,
(d) $\mathcal{S}_T$ *is a metric compactification of the ray with remainder* $\mathcal{P}$.

Theorem 9. *Let T and Q be infinite subsets of $\{n_1, n_2, \ldots\}$ such that $T \cap Q$ is finite. Then $\mathcal{S}_T$ and $\mathcal{S}_Q$ are not homeomorphic.*

Proof. Suppose to the contrary that there exists a homeomorphism $h : \mathcal{S}_T \to \mathcal{S}_Q$. Then $h(\mathcal{P}) = \mathcal{P}$ and $h(\mathcal{R}_T) = \mathcal{R}_Q$.

Suppose that $T = \{j_1, j_2, \ldots\}$ and $Q = \{k_1, k_2, \ldots\}$, where $j_1 < j_2 < \cdots$ and $k_1 < k_2 < \cdots$.

Since $\mathcal{R}_T$ is homeomorphic to $\mathcal{R}_{T'}$ for each subset T' of T of the form $T' = \{j_i, j_{i+1}, j_{i+2}, \ldots\}$ and $T \cap Q$ is finite, we may assume that $T \cap Q = \emptyset$.

For each $i \in \mathbb{N}$, let $\mathcal{R}_i^- = \bigcup\{J(k_l)\colon k_l < j_i\}$ and $\mathcal{R}_i^+ = \bigcup\{J(k_l)\colon k_l > j_i\}$. Notice that $\mathcal{R}_Q = \mathcal{R}_i^- \cup \mathcal{R}_i^+$, $\mathcal{R}_i^-$ is an arc or the empty set and, in the case that $\mathcal{R}_i^- \neq \emptyset$, then $\mathcal{R}_i^- \cap \mathcal{R}_i^+$ is a one-point set.

For each $i \in \mathbb{N}$, let $A_i = h^{-1}(\mathcal{R}_i^-) \cap J(j_i)$ and $B_i = h^{-1}(\mathcal{R}_i^+) \cap J(j_i)$. Then each one of the sets A_i and B_i is an arc and the other one is an arc, a one-point set or the empty set, and $J(j_i) = A_i \cup B_i$.

Since $\lim J(j_i) = \mathcal{P}$, $\lim J(k_l) = \mathcal{P}$ and $\operatorname{diam}(\mathcal{P}) = 1$ then there exists $M_1 \in \mathbb{N}$ such that for every $i, l \geqslant M_1$, $\operatorname{diam}(h(J(j_i)) \geqslant \frac{2}{3}$ and $\operatorname{diam}(h^{-1}(J(k_l)) \geqslant \frac{2}{3}$.

By the uniform continuity of h and h^{-1}, there exists $\delta > 0$ such that, if $\|p - q\| < \delta$, then $\|h(p) - h(q)\| < \frac{1}{8}$ and, if $\|u - v\| < \delta$, then $\|h^{-1}(u) - h^{-1}(v)\| < \frac{1}{8}$.

Let $M_2 \in \mathbb{N}$ such that $M_2 \geqslant M_1$ and, if $i, l \geqslant M_2$, then $\frac{2}{k_l}, \frac{2}{j_i} < \delta$.

Since $\lim h(J(j_i)) = \mathcal{P}$, then there exists $M_3 \in \mathbb{N}$, $M_3 \geqslant M_2$ such that, for every $i \geqslant M_3$, if $h(J(j_i)) \cap J(k_l) \neq \emptyset$, then $l \geqslant M_2$.

Claim 1. *For every $i \geqslant M_3$,* $\operatorname{diam}(A_i) \leqslant \frac{2}{3}$.

We prove Claim 1.

Proof. First, let $i \geqslant M_3$ and suppose that $l \in \mathbb{N}$ is such that the set $D = h(A_i) \cap J(k_l)$ is nondegenerate. Notice that D is a subarc of $J(k_l)$, by definition of A_i, $k_l < j_i$ and, by the choice of the sequence $n_1, n_2, \ldots,\ j_i \geqslant m_{k_l}$. Then $h^{-1} : D \to J(j_i)$ is an embedding that satisfies the conditions of Lemma 6. Since $i \geqslant M_3$ and $h(A_i) \cap J(k_l) \neq \emptyset$, then $l \geqslant M_2$. This implies that condition (b) of Lemma 6 is not satisfied. Therefore $\operatorname{diam}(h^{-1}(D)) \leqslant \frac{1}{3}$. Since $l \geqslant M_1$, we conclude that $D \neq J(k_l)$. We have shown that, if $h(A_i) \cap J(k_l)$ is nondegenerate, then $h(A_i) \cap J(k_l) \neq J(k_l)$ and $\operatorname{diam}(h^{-1}(h(A_i) \cap J(k_l))) \leqslant \frac{1}{3}$. Thus the arc $h(A_i)$ cannot contain an interval of the form $J(k_l)$ and, in fact, $h(A_i)$ is contained in a set of the form $J(k_l) \cup J(k_{l+1})$. Therefore,

$$\begin{aligned}
\operatorname{diam}(A_i) &= \operatorname{diam}\bigl(h^{-1}\bigl(h(A_i)\bigr)\bigr) = \operatorname{diam}\bigl(h^{-1}\bigl(h(A_i) \cap \bigl(J(k_l) \cup J(k_{l+1})\bigr)\bigr)\bigr) \\
&= \operatorname{diam}\bigl(h^{-1}\bigl(h(A_i) \cap J(k_l)\bigr) \cup h^{-1}\bigl(h(A_i) \cap J(k_{l+1})\bigr)\bigr) \\
&\leqslant \operatorname{diam}\bigl(h^{-1}\bigl(h(A_i) \cap J(k_l)\bigr)\bigr) + \operatorname{diam}\bigl(h^{-1}\bigl(h(A_i) \cap J(k_{l+1})\bigr)\bigr) \leqslant \frac{2}{3}.
\end{aligned}$$

This completes the proof of Claim 1. □

Claim 2. *For every* $i \geqslant M_3$, $\operatorname{diam}(h(B_i)) \leqslant \frac{2}{3}$.

Proof. Let $i \geqslant M_3$ and suppose that $l \in \mathbb{N}$ is such that the set $E = h(B_i) \cap J(k_l)$ is nondegenerate. Notice that E is a subarc of $J(k_l)$, by definition of B_i, $j_i < k_l$ and, by the choice of the sequence $n_1, n_2, \ldots, k_l \geqslant m_{j_i}$. Then $h : h^{-1}(E) \to J(k_l)$ is an embedding that satisfies the conditions of Lemma 6. Since $i \geqslant M_2$, condition (b) of Lemma 6 is not satisfied. Therefore, $\operatorname{diam}(h(h^{-1}(E))) \leqslant \frac{1}{3}$. So, $\operatorname{diam}(h(B_i) \cap J(k_l)) \leqslant \frac{1}{3}$. In particular, $h(B_i) \cap J(k_l) \neq J(k_l)$. We have shown that, if $h(B_i) \cap J(k_l)$ is nondegenerate, then $h(B_i) \cap J(k_l) \neq J(k_l)$ and $\operatorname{diam}(h(B_i) \cap J(k_l)) \leqslant \frac{1}{3}$. Thus the arc $h(B_i)$ cannot contain an interval of the form $J(k_l)$ and, in fact, $h(B_i)$ is contained in a set of the form $J(k_l) \cup J(k_{l+1})$. Therefore, $\operatorname{diam}(h(B_i)) = \operatorname{diam}(h(B_i) \cap J(k_l)) + \operatorname{diam}(h(B_i) \cap J(k_{l+1})) \leqslant \frac{2}{3}$. This completes the proof of Claim 2. □

We are ready to complete the proof of the theorem.

Let $\{A_{i_t}\}_{t=1}^{\infty}$ and $\{B_{i_t}\}_{t=1}^{\infty}$ be subsequences of $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$, respectively, such that $\lim A_{i_t} = A$ and $\lim B_{i_t} = B$, for some subcontinua A and B of $\mathcal{P}$. Since $J(j_{i_t}) = A_{i_t} \cup B_{i_t}$ and $\lim J(j_{i_t}) = \mathcal{P}$, we have that $\mathcal{P} = A \cup B$. By the indecomposability of $\mathcal{P}$, $A = \mathcal{P}$ or $B = \mathcal{P}$. By Claim 1, $\operatorname{diam}(A) \leqslant \frac{2}{3}$. Thus $B = \mathcal{P}$. So, $\lim B_{i_t} = \mathcal{P}$ and $\lim h(B_{i_t}) = h(\mathcal{P}) = \mathcal{P}$. But, by Claim 2, $\operatorname{diam}(\lim h(B_{i_t})) \leqslant \frac{2}{3}$. This contradicts the fact that $\operatorname{diam}(\mathcal{P}) \geqslant 1$ and finishes the proof of the theorem. □

Theorem 10. *There are uncountable many non-homeomorphic metric compactifications of the ray* $[0, \infty)$ *with remainder being the pseudo-arc.*

Proof. By [3, Problem 11B, p. 101], there exists an uncountable family $\{T_\lambda \colon \lambda \in \Lambda\}$ of subsets of $\{n_1, n_2, \ldots\}$ such that, for every $\lambda \neq \beta$, $T_\lambda \cap T_\beta$ is finite. Let $\Sigma = \{\mathcal{S}_{T_\lambda} \colon \lambda \in \Lambda\}$. If $\lambda \neq \beta$, then by Theorem 10, $\mathcal{S}_{T_\lambda}$ is not homeomorphic to $\mathcal{S}_{T_\beta}$. □

References

[1] M. Awantani, An uncountable collection of mutually incomparable chainable continua, Proc. Amer. Math. Soc. 118 (1993) 239–245.

[2] R.H. Bing, Concerning hereditarily indecomposable continua, Pacific J. Math. (1951) 43–51.

[3] P.R. Halmos, Problems for Mathematicians Young and Old, in: Dolciani Mathematical Expositions, Vol. 12, The Mathematical Association of America, 1991.

[4] A. Illanes, S.B. Nadler, Hyperspaces, Fundamentals and Recent Advances, in: Monographs Textbooks Pure Appl. Math., Vol. 216, Marcel Dekker, New York, 1999.

[5] W. Lewis, The pseudo-arc, Bol. Soc. Mat. Mexicana 5 (3) (1999) 25–77.

[6] V. Martínez de la Vega, The RNT property on compactifications of the ray, in: Continuum Theory, in: Lecture Notes Pure Appl. Math., Vol. 230, Marcel Dekker, New York, 2002, pp. 211–227.